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ABSTRACT

Dynamical emergent patterns of swarms are now fairly well established in nature and include flocking and rotational states. Recently, there has been great interest in engineering and physics to create artificial self-propelled agents that communicate over a network and operate with simple rules, with the goal of creating emergent self-organizing swarm patterns. In this paper, we show that when communicating networks have range dependent delays, rotational states, which are typically periodic, undergo a bifurcation and create swarm dynamics on a torus. The observed bifurcation yields additional frequencies into the dynamics, which may lead to quasi-periodic behavior of the swarm.

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Swarming behavior occurs when a large number of self-propelled agents interact using simple rules. Natural swarms of biological systems have been observed at a range of length scales forming complex emergent patterns. Engineers have drawn inspiration from these natural systems, resulting in the translation of swarm theory to communicating robotic systems. Example applications of artificial swarms include exploration and mapping, search and rescue, and distributed sensing and estimation. Through continued development, an additional parameter of delay in communication between artificial agents has become an important consideration. Specifically, it was previously discovered that communication delay will create new rotational patterns that are not observed without delay, both theoretically and experimentally. Here, we extend the understanding of communication delays to reveal the effects of range dependent delay, where the communication between agents depends on the distance between agents. The results of the research show that by including range dependent delay, new rotational states are introduced. We show how these new states emerge, discuss their stability, and discuss how they may be realized in large scale robotic systems. In improving our theoretical understanding of predicted swarm behavior modeled in simulation, we can better anticipate what will happen experimentally. Additionally, it is possible to leverage the predicted autonomous behaviors to try and force different swarm behavior.

I. INTRODUCTION

Swarming behavior, which we define as the emergence of spatiotemporal group behaviors from simple local interactions between pairs of agents, is widespread and observed over a range of application domains. Examples can be found in biological systems over a range of length scales, from aggregates of bacterial cells and dynamics of skin cells in wound healing[1–3] to dynamic patterns of fish, birds, bats, and even humans.[4–7] These systems are particularly interesting because they allow simple individual agents to achieve complex tasks in ways that are scalable, extensible, and robust to failures of individual agents. In addition, these swarming behaviors are able to form and persist in spite of complicating factors such as delayed actuation, latent communication, localized number of neighbors each agent is able to interact with, heterogeneity in agent dynamics, and environmental noise. These factors have been the focus of previous theoretical research in describing the bifurcating spatial-temporal patterns in swarms, as seen, for example, in Refs. 8–11. Likewise, the application of swarms has been experimentally realized in areas such as mapping,[12] leader-following,[13,14] and density control.[15] To guarantee swarming behavior experimentally, control is typically employed[16–20] to prove convergence to a given state by relying on strict assumptions to guarantee the desired behavior. However, by relaxing certain assumptions, a number of studies show that even with simple interaction protocols, swarms of agents are able to converge to organized, coherent behaviors in a self-emergent manner, i.e., autonomously without control. Different mathematical approaches yielded a wide selection of both agent-based[4,5,7,21] and continuum models that predict swarming dynamics.[2,8,22] In almost all models, since the agents have just a few simple rules, there exists only a relatively small number of controllable parameters. The parameter set usually consists of a self-propulsion force, a potential function governing attracting and repelling forces between agents, and a communicating radius governing the local neighborhood at which the agents can sense and interact with each other.

In both robotic and biological swarms, an additional parameter appears as a delay between the time information is perceived and the actuation (reaction) time of an agent. Such delays have now been measured in swarms of bats, birds, fish, and crowds of people.[23–25] The measured delays are longer than the typical relaxation times of the agents and may be space and time dependent. Robotic swarms experience communication delays, which provide similar effects to the delay experienced in natural swarms. Incorporating stationary delays along with a minimal set of parameters in swarm models results in multi-stability of rotational patterns in space.[26–30] In particular, for delays that are equal and fixed, one observes three basic swarming states or modes: Flocking, which is a translating center of mass; ring state, where the agents are splayed out on a ring in phase about a stationary center of mass; and rotating state, where the center of mass itself rotates.

Synthetic robotic swarms have communication delays that naturally occur over wireless networks as a result of low bandwidth[31] resulting in delayed communication and multi-hop communication.[32] In cases where the delays are fixed and equal and the communication occurs on a homogeneous network, it is known that delays create new rotational patterns, as has been verified both theoretically and experimentally.[27,28] However, in situations with robots, even simple communication models are based on the distance between agents.[33,34]

Following from these models, if one assumes that the delays are range dependent, the problem becomes one of studying state dependent delays where delays depend implicitly on the relative positions between agents.

When placing swarms in realistic complex environments, delays are not necessarily a continuous function of range, but rather, it is the increasing probability of delays increasing stochastically when agents move further away from one another beyond a certain radius.[35,36] That is, the rate of communication becomes spatially dependent, whereby near agents see a signal with a fast rate of communication, but due to shading and fading of signals, communication rates are slowed and complex outside a given radius. Underwater communication is an excellent swarm example where delays outside a significant radius impart rates of communication of one to two orders of magnitude greater than local communication rates.[37]

The swarm model that follows takes a globally coupled swarm and explicitly relaxes the fixed delay assumption by including range dependent delay based on a fixed communication radius. We show that when range dependent delays are included, new frequencies are introduced and generate bifurcations to a torus. The result is a milling type of swarm that depends on just a few parameters. The results here are important for robotic swarming where one of the goals is to produce desired patterns autonomously, without external controls. The pattern formations predicted here show how delayed information, whether coming from communication, actuations, or both, impacts the stability of swarm states, such as ring and/or rotating states. By revealing those parameter regions where patterns are destabilized, we provide a comprehensive characterization of the autonomously accessible swarm states in the presence of range dependent delay.

II. THE SWARM MODEL

Consider a swarm of delay-coupled agents in $\boldsymbol{R}^2$. Each agent is indexed by $i \in \{1, \ldots, N\}$. We use a simple but general model for swarming motion. Each agent has a self-propulsion force that strives to maintain motion at a preferred speed and a coupling force that governs its interaction with other agents in the swarm. The interaction force is defined as the negative gradient of a pairwise interaction potential $U(\cdot,\cdot)$. All agents follow the same rules of motion; however, mechanical differences between agents may lead to heterogeneous dynamics; this effect is captured by assigning different acceleration factors (denoted κ_i) to the agents. In this paper, we assume $\kappa_i = 1$ for all i. For the effect of heterogeneity on the swarm bifurcations, see Ref. 9.

Agent-to-agent interactions occur along a graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, where $\mathscr{V}$ is the set of vertexes v_i in the graph and $\mathscr{E}$ is the set of edges e_{ij}. The vertices correspond to individual swarm agents, and edges represent communication links, that is, agents i and j communicate with each other if and only if $e_{ij} \in \mathscr{E}$. All communication links are assumed to be bi-directional, and all communications occur with a time delay τ. That is, range dependence is not included. Let $\mathbf{r}_i \in \mathbb{R}^2$ denote the position of the agent i and let $\mathscr{N}_i = \{v_j \in \mathscr{V} : e_{ij} \in \mathscr{E}\}$ denote its set of neighbors of agent i. The motion of agent i is governed by the following equation:

$$\ddot{\mathbf{r}}_i = \kappa_i(1 - \|\dot{\mathbf{r}}_i\|^2)\dot{\mathbf{r}}_i - \kappa_i \sum_{j \in \mathscr{N}_i} \nabla_x U(\mathbf{r}_i(t), \mathbf{r}_j^{\tau}(t)), \tag{1}$$

where superscript τ is used to denote the time delay so that $\mathbf{r}_j^{\tau}(t) = \mathbf{r}_j(t-\tau)$, $\|\cdot\|$ denotes the Euclidean norm, and ∇_x denotes the gradient with respect to the first argument of U. The first term in Eq. (1) governs self-propulsion, where the speed has been normalized to unity, that is, without coupling the agents always asymptote to unit speed.

To analyze the dynamics of a large scale swarm, we use a harmonic interaction potential with short-range repulsion,

$$U(\mathbf{r}_i, \mathbf{r}_j^\tau) = c_r e^{-\frac{\|\mathbf{r}_i - \mathbf{r}_j\|}{l_r}} + \frac{a}{2N}\left\|\mathbf{r}_i - \mathbf{r}_j^\tau\right\|^2. \quad (2)$$

In Eq. (1), it is assumed that the communication delay, τ, is independent of the distance, or range, between any pair of agents. (Notice that the exponent of the repulsion term is independent of the delay since the repulsion force is local.) With the addition of delays in the network, it was shown in homogeneous communication networks that in addition to the usual dynamical translating and milling (or ring) states, for sufficiently large τ, new rotational states emerge.[27] In particular, for a given attractive coupling strength, there is a delay that destabilizes the periodic ring state into a rotating state, in which the agents coalesce to a small group and move around a fixed center of rotation; this behavior is quite different from the ring state where agents are spread out in a splay state phase. The rotating state is only observed with delay introduced in the communication network, and it appears through a Hopf bifurcation.

However, in real-world robotic swarms, communication delays are not uniform between all pairs of agents; delays may be stochastic or even state dependent. For example, if agents are communicating over a multi-hop network, the delay will increase with the number of hops required to send a message from one agent to the other and, in general, will scale with the separation between them. In order to handle range dependent delays, we will make an approximation that depends on a communication range radius.

A. Approximating range dependent delayed coupling

For the coupling term, we are interested in introducing an approximation to range based coupling delay. Since all communicating agents send signals with some delay, we compute relative distances defined as

$$D_{i,j}^\tau \equiv ||\mathbf{r}_i - \mathbf{r}_j^\tau||. \quad (3)$$

We define a Heaviside function, $H(x)$, that is, zero when $x \le 0$ and 1 otherwise, and we employ global coupling based on a spring potential. For our range dependent metric, we let $\epsilon \ge 0$ denote the range radius. Suppose that when the separation between two agents is small, that is, less than ϵ, then sensing between two agents is almost immediate. In practice, the time needed for sensing depends on several factors, such as actuation times, and so distances in practice are computed with delay. Therefore, we model the coupling term for the ith agent as

$$\begin{aligned} C_i(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_j^\tau, \epsilon) = &-\frac{a}{N}(\nabla_x U(\mathbf{r}_i(t), \mathbf{r}_j^\tau(t)))H(D_{i,j}^\tau - \epsilon) \\ &-\frac{a}{N}(\nabla_x U(\mathbf{r}_i(t), \mathbf{r}_j(t)))(1 - H(D_{i,j}^\tau - \epsilon)), \end{aligned} \quad (4)$$

where the first coupling term has the delay turned on since the distance is outside a ball of radius ϵ, while the second term has no delay since the distance is within the ϵ ball. The resulting swarm model with range dependence from Eq. (4) is now

$$\ddot{\mathbf{r}}_i = \kappa_i(1 - \|\dot{\mathbf{r}}_i\|^2)\dot{\mathbf{r}}_i - \kappa_i \sum_{j \in \mathcal{N}_i} C_i(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_j^\tau, \epsilon). \quad (5)$$

If the delayed distance is within an ϵ ball, then we evaluate the coupling without delay. Otherwise, the coupling is delayed. Thus, the coupling function takes into account when delay is active or not between pairs of communicating agents and depends on the range radius, ϵ.

The Heaviside function of the right hand side of Eq. (9) renders the differential delay equation derivatives discontinuous, and as such poses a numerical integration problem. To mollify the lack of smoothness, we approximate $H(x)$ by letting $H(x) \approx \frac{1}{\pi}\arctan(kx) + \frac{1}{2}$, where $k \gg 1$ and constant, and limits on the Heaviside function as $k \to \infty$.

Using only the delayed distance to compute a range dependent coupling assumes that any measurement is not instantaneous. If one were to be able to compute the ideal situation where delay would not be a sensing factor, then certain issues would need to be resolved, which we do not consider here.

B. Numerical simulations of full swarms

Examples of simulations using the swarm model with the range dependent coupling are shown below. Here, the number of agents $N = 150$, and the coupling strength $a = 2.0$. For the remainder of the analysis, we set $c_r = 0$, and note that the attractors persist when the repulsive amplitude is sufficiently small[27] (see the supplementary material for a video of the dynamics with small repulsion).

Note that even when ϵ is very small, as shown in Fig. 1, we observe a mix of clustered states which are a combination of pure ring and rotation states. The agents tend to cluster into local groups, and the clusters move in clockwise and counterclockwise directions as in the ring state. Here, however, the phase differences between agents are non-uniform. When examining a single random agent, as shown in Fig. 2, it is periodic with a sharp frequency of rotation, and the relative positions of all individual agents are phase locked.

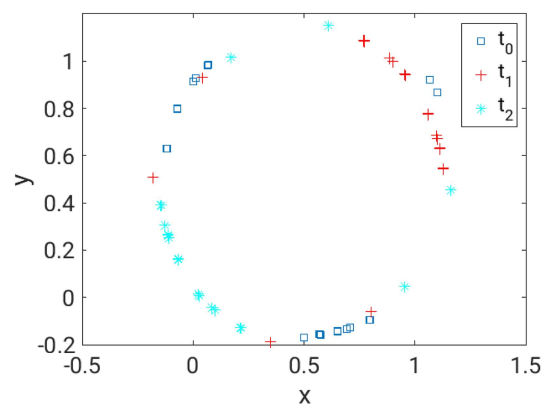

FIG. 1. Three snapshots of swarm state in space for $\epsilon = 0.01$, $a = 2.0$, and $\tau = 1.75$. Sample times t_0, $t_1 = t_0 + 20$, and $t_2 = t_0 + 40$.

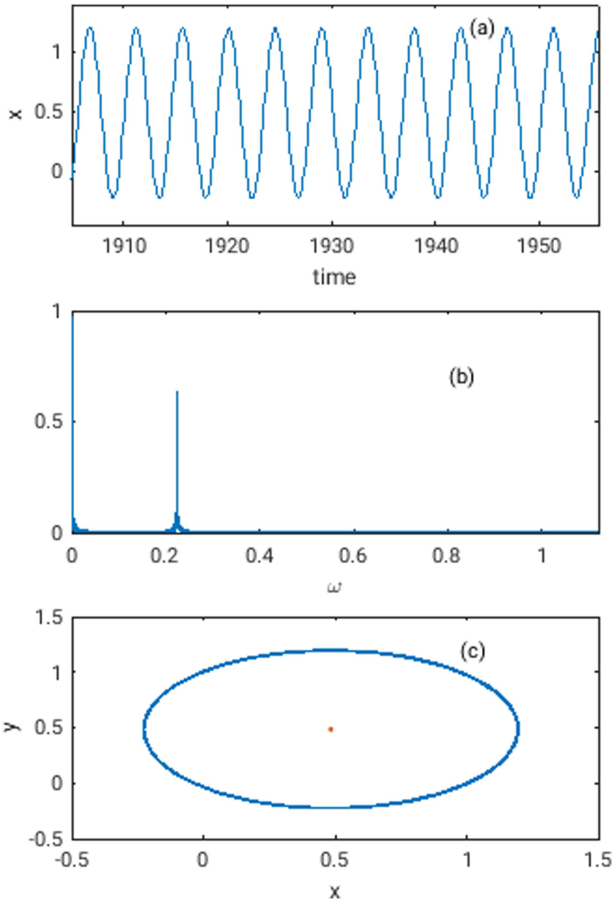

FIG. 2. Swarm ring state for $\epsilon = 0.01$, $a = 2.0$, and $\tau = 1.75$. (a) Time series of the x-component of a single agent. (b) The power spectrum showing a sharp frequency. (c) A phase portrait of the orbit of a single agent. The red point denotes the center of mass.

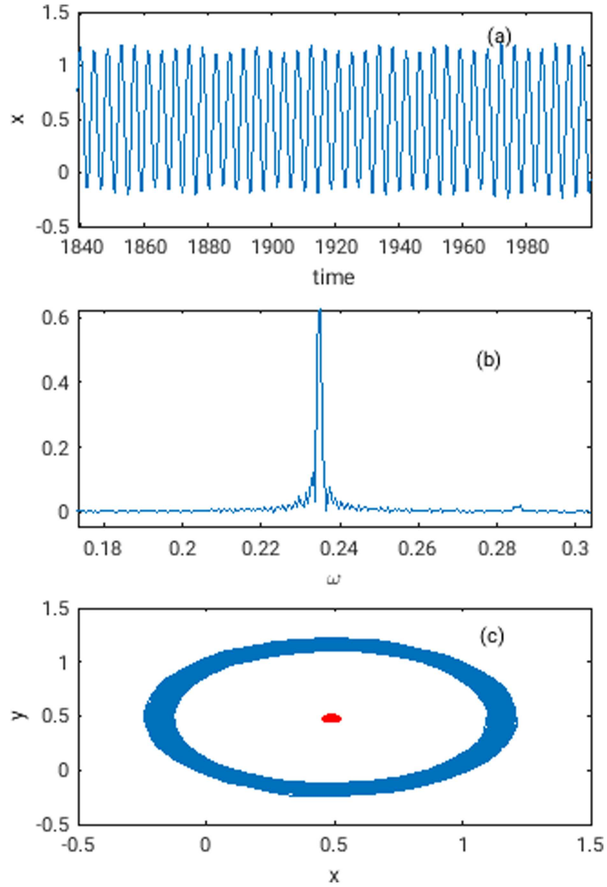

FIG. 3. Swam instability $\epsilon = 0.25$, $a = 2.0$, and $\tau = 1.75$. (a) Time series of the x-component of a single agent. (b) The power spectrum showing a slight broadening and birth of a new frequency. (c) A phase portrait of the orbit of a single agent.

When considering the center of mass of the positions over all agents, $\boldsymbol{R} \equiv \frac{1}{N}\sum_i \boldsymbol{r}_i$, the center of mass does small amplitude oscillations about a fixed point (not shown).

As the radius ϵ increases, instability of the periodic mixed state occurs, giving rise to more complicated behavior, as seen in Fig. 3. New frequencies are introduced, causing the ring state to appear as a quasi-periodic attractor. Moreover, the dynamics of the center of mass has its own non-trivial dynamics which includes the effects of new frequencies. By examining the Poincare map of the attractors, the instability gives rise to dynamics, which we conjecture is motion on a torus. Letting (M_x, M_y) denote the time averaged center of mass over all agents, we compute the sequence $x(t_i), i = 1 \ldots M$ when $y(t_i) = 0$ and $x(t_i) > M_x$. The result is shown in the two panels in Fig. 4. Panel (a) shows a complicated toroidal motion after transients are removed of the center of mass in Fig. 3(c). For a single frequency, the dynamics of the center of mass would be a single fixed point. The addition of new frequencies is revealed in the Poincare map as complicated motion on a torus. For larger values of ϵ, the motion on the torus converges to a periodic attractor in panel (b).

III. MEAN FIELD EQUATION OF RANGE DEPENDENT DELAY-COUPLED SWARM

In order to shed some light on the origin of the bifurcation to dynamics on a torus, we examine the full swarm model from a mean field perspective. The mean field is much lower dimensional, and a full bifurcation analysis may be done. We consider the case of all-to-all communication. Let

$$\boldsymbol{R} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}_i$$

and

$$\mathbf{r}_i = \boldsymbol{R} + \delta\mathbf{r}_i,$$

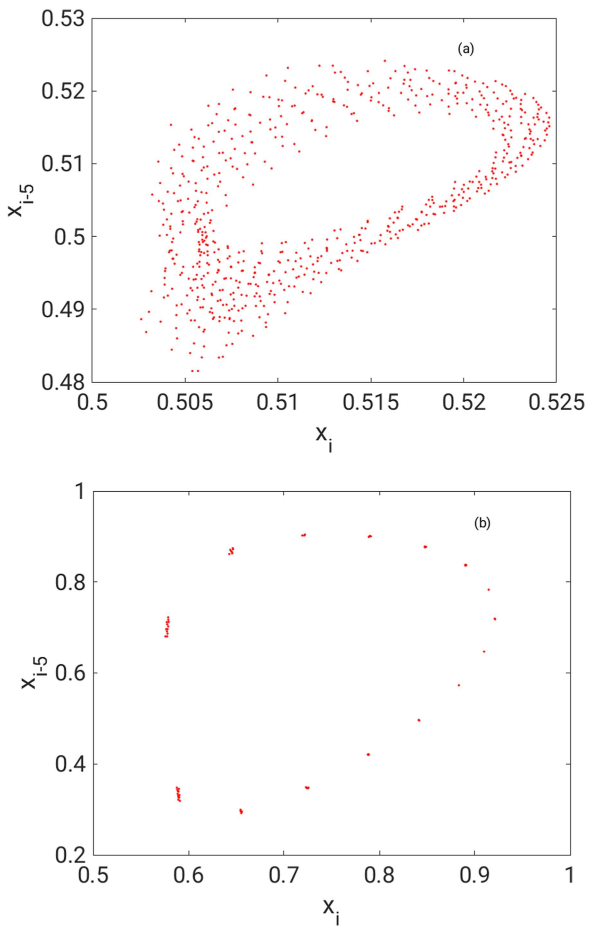

FIG. 4. Poincare map of Eqs. (1)–(4) for (a) $\epsilon = 0.25$ and (b) $\epsilon = 0.5$. Other parameters are fixed: $a = 2.0$ and $\tau = 1.75$. See text for details.

where $\delta \mathbf{r}_i$ is a fluctuation term with the identity, and

$$\sum_{i=1}^{N} \delta \mathbf{r}_i = 0. \tag{6}$$

Then, we can write Eq. (5) as

$$\begin{aligned} \ddot{\boldsymbol{R}} + \delta\ddot{\mathbf{r}}_i = {} & (1 - |\dot{\boldsymbol{R}} + \delta\dot{\mathbf{r}}_i|^2)(\dot{\boldsymbol{R}} + \delta\dot{\mathbf{r}}_i) \\ & - \frac{a}{N} \sum_{j=1, j\neq i}^{N} ((\boldsymbol{R} + \delta\mathbf{r}_i) - (\boldsymbol{R}^\tau + \delta\mathbf{r}_j^\tau))C_{1,i} \\ & - \frac{a}{N} \sum_{j=1, j\neq i}^{N} ((\boldsymbol{R} + \delta\mathbf{r}_i) - (\boldsymbol{R} + \delta\mathbf{r}_j))C_{2,i}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} C_{1,i} &= H(\|\mathbf{r}_i - \mathbf{r}_j^\tau\| - \epsilon) \\ &= H(\|(\boldsymbol{R} + \delta\mathbf{r}_i) - (\boldsymbol{R}^\tau + \delta\mathbf{r}_j^\tau)\| - \epsilon) \\ &= H(\|\boldsymbol{R} - \boldsymbol{R}^\tau + \delta\mathbf{r}_i - \delta\mathbf{r}_j^\tau\| - \epsilon) \end{aligned}$$

and

$$C_{2,i} = 1 - C_{1,i}.$$

We use the following to reduce the equations of motion to the mean field: from Eq. (6), we note

$$\sum_{i=1}^{N} \delta\mathbf{r}_i^\tau = \sum_{j=1, j\neq i}^{N} \delta\mathbf{r}_j^\tau + \delta\mathbf{r}_i^\tau = 0 \iff - \sum_{j=1, j\neq i}^{N} \delta\mathbf{r}_j^\tau = \delta\mathbf{r}_i^\tau. \tag{8}$$

We further assume that all perturbations from the mean, $\delta\mathbf{r_i}$, are all negligible (this is always true if the coupling amplitude is sufficiently large). In addition, we use the fact that $\frac{a(N-1)}{N}$ limits to a, as $N \to \infty$. Therefore, we obtain mean field approximation for the center of mass of range dependent coupled delay case,

$$\ddot{\boldsymbol{R}} = (1 - |\dot{\boldsymbol{R}}|^2) \cdot \dot{\boldsymbol{R}} - a(\boldsymbol{R} - \boldsymbol{R}^\tau) \cdot H(\|\boldsymbol{R} - \boldsymbol{R}^\tau\| - \epsilon). \tag{9}$$

IV. NUMERICAL ANALYSIS OF THE MEAN FIELD EQUATION

A. Examples of rotational attractors

As in the case for the full multi-agent system, we see the existence of periodic behavior for τ sufficiently below an instability threshold, as shown in the time series of Fig. 5. As we increase τ, we

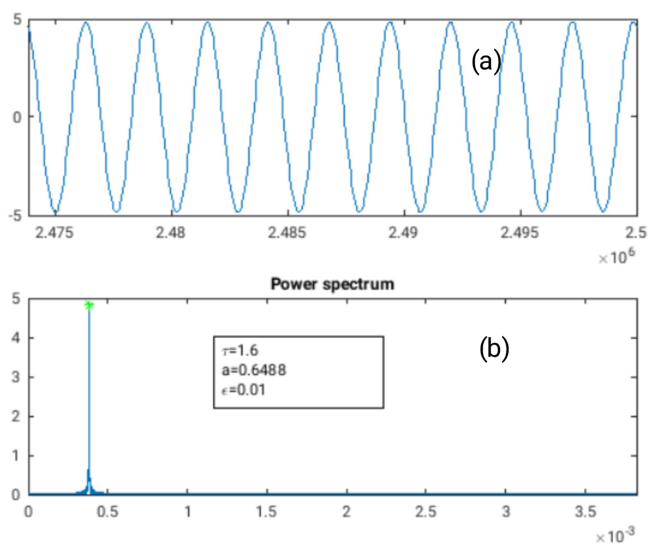

FIG. 5. Periodic motion of the mean field equation (9) for $\epsilon = 0.01$, $a = 0.64$, and $\tau = 1.6$. (a) Time series of the x-component of the mean field. (b) Power spectra of the time series.

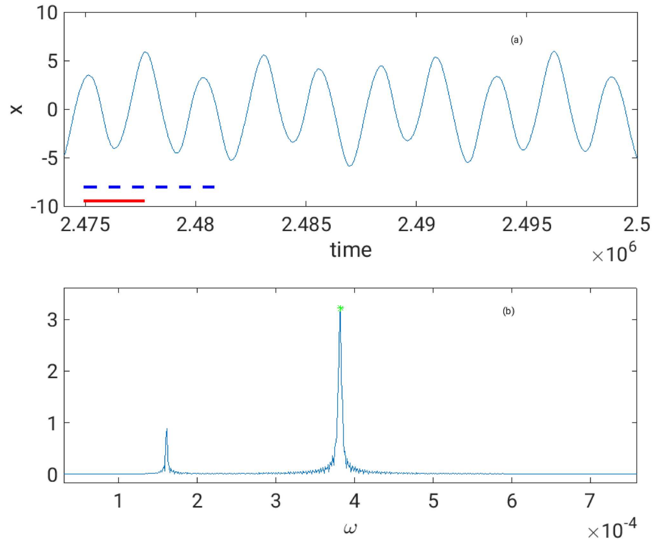

FIG. 6. Quasi-periodic motion of the mean field equation (9). (a) Time series of the x-component of the mean field. Solid (red) line denotes period length of dominant spectral peak. Dashed line denotes period length of secondary peak. (b) Power spectra of the time series.

expect the periodic orbit to lose stability, resulting in a new attractor. In particular, one notices the emergence of a new frequency in addition to the existing dominant one, as shown in Fig. 6 The additional frequency usually implies a bifurcation to dynamics on a torus or a higher dimensional torus.

We now investigate this transition by tracking the stability via monitoring the Floquet exponents corresponding to the periodic orbit. For a general differential delay equation given by $\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}(t), \boldsymbol{x}(t-\tau))$, if $\boldsymbol{\phi}(t) = \boldsymbol{\phi}(t+T)$ for all $t \geq 0$, then stability is determined by examining the linearized equation along $\boldsymbol{\phi}(t)$,

$$\dot{\boldsymbol{X}}(t) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}(t)}(\boldsymbol{\phi}(t), \boldsymbol{\phi}(t-\tau))\boldsymbol{X}(t) + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}(t-\tau)}(\boldsymbol{\phi}(t), \boldsymbol{\phi}(t-\tau))\boldsymbol{X}(t-\tau). \tag{10}$$

The stability of the periodic solution is determined by the spectrum of the time integration operator $U(T, 0)$, which integrates Eq. (10) around $\phi(t)$ from time $t = 0$ to $t = T$. This operator is called the monodromy operator and its (infinite number of) eigenvalues, which are independent of the initial state, are called the Floquet multipliers.[38] For autonomous systems, it is necessary and sufficient there exists a trivial Floquet multiplier at 1, corresponding to a perturbation along the periodic solution.[39,40] The periodic solution is stable provided all multipliers (except the trivial one) have modulus smaller than 1; it is unstable if there exists a multiplier with modulus larger than 1. Bifurcations occur whenever Floquet multipliers move into or out of the unit circle. Generically, three types of bifurcations occur in a one parameter continuation of periodic solutions: a turning point, a period doubling, and a torus bifurcation where a branch

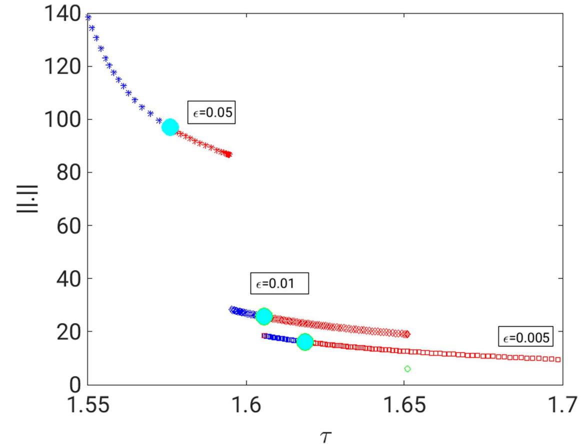

FIG. 7. Bifurcation plot showing the norm of the periodic orbits as a function of delay τ. Parameter a=0.68. Red (blue) markers denote unstable (stable) orbits. Cyan symbols denote the change in stability where a pair of complex eigenvalues cross the imaginary axis.

of quasi-periodic solutions originates and where a complex pair of multipliers crosses the unit circle.[38]

We have tracked a set of stable periodic orbits for various radii of ϵ and located the change in stability by computing the Floquet multipliers. The results plotted in Fig. 7 show that for a range of radii ϵ, there exists a bifurcation to a torus at some delay. Notice that as ϵ increases, there results an increase in the size of the orbits, which qualitatively agrees with our full agent based simulations.

Since there exists a range of delays that destabilize periodic swarm dynamics for each ϵ, we summarize the onset of torus bifurcations by plotting the locus of points at which stability changes as

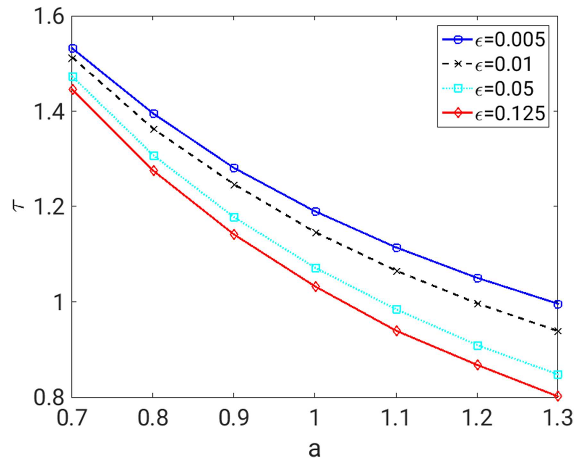

FIG. 8. Plotted is the locus of points at which torus bifurcations emerge as a function of coupling amplitude a, delay τ for various range radii ϵ for the mean field equation (9).

a function of coupling amplitude and delay. The results are plotted in Fig. 8.

Figure 8 is revealing in that it shows a functional relationship of the bifurcation onset that is similar over a range of ϵ. For larger values of ϵ, it is clear that lower values of delay and coupling are required to generate bifurcations. This holds true over two orders of ϵ. For a fixed value of ϵ, we also see monotonic relationship between delay and coupling strength, so that it is easier for smaller delays to destabilize periodic motion for larger coupling strengths.

V. CONCLUSIONS

We considered a new model of a swarm with delay-coupled communication network, where the delay is considered to be range dependent. That is, given a range radius, delay is on if two agents are outside the radius and zero otherwise. The implication is that small delays do not matter if the agents are close to each other.

The additional range dependence creates a new set of bifurcations not previously seen. For general swarms without delay, the usual states consist of flocking (translation) or ring/rotational state (milling), with agents spread in phase. With the addition of a fixed delay, a rotational state bifurcates that has all agents in phase and rotate together.[41] Range dependence introduces a new rotational bifurcating state that exhibits behavior observed as a new mixed state combining dynamics of both ring and rotating states.

The radius parameter, ϵ, was used to quantify the bifurcation of the rotational mixed state. For small ϵ, we see that the dynamics for the full swarm shows clustered counter-rotational behavior that is periodic. This agrees for small radius values in the mean field description as well. As the radius increases, the mixed periodic state generates new frequencies in the full model, which are manifested as torus bifurcations in the mean field. Mean field analysis was done by tracking Floquet multipliers that cross the imaginary axis as complex pairs. Frequency analysis explicitly shows the additional frequencies in the mean field.

Finally, we tracked the locus of coupling amplitudes and delay for various values of ϵ locating the parameters at which torus bifurcation occur. The results reveal that as ϵ increases, torus bifurcations onset at lower values of coupling amplitude and delay. The implications are that more complicated behavior than periodic motion has a greater probability of being observed in both theory and experiment if range dependence of delay is included.

SUPPLEMENTARY MATERIAL

The videos in the supplementary material show the attractor of a swarm consisting of $N = 300$ agents. Fixed parameters for the three videos are $a = 2.0$ and $\tau = 1.75$. The parameters for zero radius (delay is on all the time) are $\epsilon = 0.0, c_r = 0.05$, and $l_r = 0.05$ for a baseline, as shown in Video1_eps_0p0.mp4. The parameters corresponding to Fig. 2 are $\epsilon = 0.01, c_r = 0.01$, and $l_r = 0.05$, as shown in Video2_eps_0p01.mp4. The video shows that the attractor persists when repulsive forces are local and weak. Similar behavior is observed when N=150, which is used in Fig. 1 without repulsion, i.e., $c_r = 0$. The parameters corresponding to Fig. 3 are $\epsilon = 0.25$, $c_r = 0.05$, and $l_r = 0.05$, as shown in Video3_eps_0p25.mp4.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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