Using Control to Shape Stochastic Escape and Switching Dynamics

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(Dated: 24 January 2019)

We present a strategy to control the mean stochastic switching times of general dynamical systems with multiple equilibrium states subject to Gaussian white noise. The control can either enhance or abate the probability of escape from the deterministic region of attraction of a stable equilibrium in the presence of external noise. We synthesize a feedback control strategy that actively changes the system’s mean stochastic switching behavior based on the system’s distance to the boundary of the attracting region. With the proposed controller, we are able to achieve a desired mean switching time, even when the strength of noise in the system is not known. The control method is analytically validated using a one-dimensional system and its effectiveness is numerically demonstrated for a set of dynamical systems of practical importance.

PACS numbers: Valid PACS appear here
Keywords: Controlling noise-induced transitions, Controlling mean switching times, Controlling rare events, Large deviation theory

Noise is an inherent phenomena in all physical dynamical system. Thus, the behaviour of a dynamical system under the influence of noise has been a widely studied topic. In particular, the effect of small noise on the stability of a system has generated significant attention in the literature. It has been shown that under the influence of noise, a system can be made to transition out of the its deterministically stable states. While these are rare occurrences for small noise, they have a significant impact on the overall behaviour of the system. The expected time for such a transition to occur, i.e., the mean switching time, is an important characteristic in such systems. In this work, we show how an external control could be used to enhance or abate this switching behavior, and synthesize a feedback control strategy that actively changes the mean switching time to a desired value. This enables one to control the dwell time of the system in a given basin of attraction, to a desired value. We analyse the controller using a representative one-dimensional system and demonstrate the controller on a set of dynamical systems with practical importance.

I. INTRODUCTION

The trajectories of a deterministic dynamical system is completely defined by the initial conditions. With initial conditions in the region of attraction of a stable equilibrium, the system is expected to approach the equilibrium state and remain there indefinitely. One would expect this deterministic behavior to be only slightly altered in the presence of small noise, since switching between stable states would require the system to overcome a large activation barrier. However, one often sees these types of rare, noise-induced switching events in a variety of physical and biological systems. A few examples of phenomena that exhibit rare transition events include extinction of disease\textsuperscript{1,2} or species\textsuperscript{3}, switching between gene states\textsuperscript{4} or magnetization states\textsuperscript{5}, and transitions in ocean flows\textsuperscript{6,7}.

In the presence of noise, the system trajectories are no longer prescribed by the initial conditions. Instead, the behavior of the system is now described by the probability density which indicates the likelihood of achieving a particular system state. With this viewpoint, metastable equilibria will be peaks in this probability landscape. One important feature of interest when studying noise-induced transitions is the optimal escape pathway from a metastable state or the optimal transition pathway from one metastable state to another. Of the many paths that lead to escape from a metastable state, or switching between two such states, there exists a most probable transition path. This path is known as the optimal escape or switching path. It is of great importance in a variety of applied problems to determine this optimal path since knowledge of the path then enables the determination of the mean time to escape from a metastable state or to switch from one metastable state to another.

While the noise that induces these rare transition events may be internal or external to the system, in this article we only consider external noise. Mathematically, the effect of external noise is often described using a Langevin equation or the associated Fokker-Planck equation (though the dynamics of external noise may sometimes be described by a master equation\textsuperscript{8}). Feynman famously pointed out that each noise realization corresponds to a particular trajectory of the system, and therefore the probability density of realizations of trajectories is determined by the probability density of noise

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Controlling Stochastic Transition Rates

The mean switching time (MST) essentially describes the dwell time of the system in a given basin of attraction, and as such it is an important characteristic of the system. Accordingly, there has been an increased interest in the literature on using an external control to abate or enhance the mean switching time\textsuperscript{6,12,13}. However, the control strategies proposed in those works cannot control the switching time to a desired value. This is in part due to the difficulty of obtaining a closed form expression for the MST with fully described parameters, whose computation does not require experimentation or simulation. Such analytical expressions are only available for a very limited set of low-dimensional systems, e.g., escape from a one-dimensional potential well\textsuperscript{14}.

In contrast, in the current work we present a control strategy which uses limited control to achieve a desired MST. The strategy relies on knowledge about the basin boundaries of the region of attraction of the stable equilibrium of the deterministic system. These basin boundaries can be obtained using a variety of methods, including finite-time Lyapunov exponent (FTLE) computations\textsuperscript{15}. The required actuation for the strategy is minimal since the actual transition is precipitated by the noise in the system. We developed the approach in our previous work\textsuperscript{16}, where we exploited noise-driven transitions to control the dwell time of a marine robot operating in a gyre flow. In this work, we generalize the approach to a broader class of dynamical systems. The method enables one to control the MST of a stochastic dynamical system with metastable states to a desired value. To demonstrate the generality of the approach, we evaluate our method on two dynamical systems: 1) a double-gyre flow field, and 2) a damped pendulum. The double-gyre flow is often used to model large scale circulations in the ocean\textsuperscript{17}, while the damped pendulum is representative of many practical dynamical systems, e.g., phase difference across a Josephson junction\textsuperscript{18}. To the best of our knowledge, this is the first attempt at using control to obtain a desired MST.

The rest of the article is organized as follows. In section II the background of the stochastic switching problem is presented while the effect of an external control field on the MST is analyzed in section III. The proposed control strategy is presented in section IV, and validation of the strategy for different systems is presented in section V. The article contains concluding remarks in section VI.

II. BACKGROUND

We consider a dynamical system that is affected by external noise. The system is modeled using the Langevin equation

\[ \dot{x} = F(x) + \eta(t), \quad (1) \]

where \( x \in \mathbb{R}^n \) is the state variable, \( F \) is the nominal system, and \( \eta(t) \) is an uncorrelated white noise term where each component has zero mean and a standard deviation of \( \sigma = \sqrt{2D} \), where \( D \) is the noise intensity. In (1), \( F \) describes a deterministic nonlinear system with multiple equilibrium states. In addition to modelling errors, \( \eta(t) \) can also capture sensing, actuation, and environmental uncertainties. In the absence of noise, all trajectories of the system will approach the stable equilibrium states of the system. With the addition of external noise, the system trajectories will be governed by individual noise realizations. In fact, each realization of the noise \( \eta \) results in a corresponding trajectory of the state variable \( x \). The trajectories will now be concentrated around the metastable equilibria, and the probability density of the trajectories over the state space will have peaks near these equilibria. Even with infinitesimally small noise, there are rare noise-induced events in which the system transitions from one metastable state to another.

Since each realization of the noise \( \eta \) results in a corresponding trajectory of the state variable \( x \), the probability of occurrence of a switch from one metastable state to another is governed by the probability of occurrence of the corresponding noise realization \( \eta(t) \). Of all the possible escape trajectories from a metastable state, there exists a trajectory that is probabilistically most likely to occur. It has been shown\textsuperscript{10,19} that the probability \( P \) of occurrence of a given noise trajectory is

\[ P \propto e^{-\mathcal{R}/D}, \quad (2) \]

where \( R \) is the action and is given as

\[ R = \frac{1}{2} \int_{t_0}^{t_f} \eta(t)^T \eta(t) \, dt = \frac{1}{2} \int_{t_0}^{t_f} [\dot{x} - F(x)]^T [\dot{x} - F(x)] \, dt. \quad (3) \]

Thus the most probable switching path is the one with the minimum action given by

\[ \mathcal{R} = \min_{x(t)} \left\{ \frac{1}{2} \int_{-\infty}^{\infty} [\dot{x} - F(x)]^T [\dot{x} - F(x)] \, dt \right\}. \quad (4) \]

Given an optimal path, (1) can be used to find the associated optimal noise realization. In the case of small noise, the switching rate is directly proportional to the probability of observing this optimal, or most probable, noise profile as all other noise realizations are exponentially less likely to occur\textsuperscript{20}. The mean switching time (MST) can therefore be approximated by

\[ T_E = be^{\mathcal{R}/D}, \quad (5) \]
FIG. 1. (a) Phase portrait of the double-gyre flow for \( A = 1, \ s = 1 \) and \( \mu = 1 \). The black cross indicates the stable equilibrium at the center of the left gyre, and the red cross indicates the saddle point at the lower right corner; (b) Logarithm of the MST \( T_E \) versus \( 1/D \) for the double-gyre flow obtained using Monte Carlo simulations. The simulation results are consistent with the form of the theoretically predicted relationship between the MST and noise intensity given in (5). Red crosses are simulation data points, and the solid blue line is the line of best fit.

where \( b \) is a prefactor determined through numerical simulation or an experiment.

Figure 1a shows the phase portrait of a double-gyre flow field given by

\[
F(x, y) = \begin{bmatrix}
F_x(x, y) \\
F_y(x, y)
\end{bmatrix} = \left[ -\pi A \sin(\frac{\pi x}{s}) \cos(\frac{\pi y}{s}) - \mu x \\
\pi A \cos(\frac{\pi x}{s}) \sin(\frac{\pi y}{s}) - \mu y
\right],
\]

where \( A \) denotes the strength of the flow, \( s \) is a scaling factor for the gyre dimensions, and \( \mu \) is a damping coefficient. Figure 1b shows the mean switching times obtained by performing Monte Carlo simulations for a range of noise intensities in this flow field for \( A = 1, \ s = 1 \) and \( \mu = 1 \). In these simulations, the system is initialized by placing a particle/sensor near the metastable state at the center of the left gyre. The particle will stay in the left gyre for a long period of time, but eventually the noise will cause the particle to undergo an escape event. The escape from the left gyre to the right gyre occurs when the particle transitions across the gyre boundary demarcated by the stable and unstable manifolds of the saddle points flanking each gyre. In Fig. 1b, the crosses in red indicate simulation data points, and the solid line in blue indicates the line of best fit. It can be seen that the simulation results are consistent with the the form of (5), and the intercept of this line of best fit allows us to obtain an estimate of the prefactor \( b \).

FIG. 2. The most probable switching path (solid black curve) plotted on top of the phase portrait of (a) a double-gyre flow field, and (b) a damped pendulum. The red line shows the boundary of the attracting region. The MPSP is computed from a stable equilibrium to one of the saddle points on the boundary, and it is truncated near the stable equilibrium for clarity. The optimal noise profile, shown in blue arrows along the MPSP, is directed towards the nearest boundary.

III. STOCHASTIC TRANSITIONS WITH CONTROL

The central theme of this work is the control of the MST using an external control signal. The controlled system dynamics are given by

\[
\dot{x} = F(x) + u(x, t) + \eta(t),
\]

where \( u \) is the control signal. Inspired by the noise profile associated with the optimal switching path (see Fig. 2), a control signal of the form \( u = cf(x) \) is considered. The function \( f(x) \) gives the direction of the control with \( ||f(x)|| = 1 \), and similar to the most probable noise profile, \( f(x) \) is selected to point towards the closest basin boundary. Using (4), the action of the trajectory that is most likely to result in escape for this controlled system is given by,

\[
\mathcal{R}^c = \min_{\text{x}(t)} \frac{1}{2} \int_{-\infty}^{\infty} [\dot{x} - F(x) - cf(x)]^T [\dot{x} - F(x) - cf(x)] dt.
\]

Let the optimal switching path that is the solution to (9) be denoted by \( x^c(t) \). Thus, the action of the most likely noise profile can be re-written as

\[
\mathcal{R}^c = \frac{1}{2} \int_{-\infty}^{\infty} [\dot{x}^c - F(x^c) - cf(x^c)]^T [\dot{x}^c - F(x^c) - cf(x^c)] dt.
\]

When \( c = 0 \), the action \( \mathcal{R}^0 \) is given by the solution to the uncontrolled case in (4), and the corresponding MPSP is
\( x^0(t) \). Note that, for an arbitrary \( c \), the optimal path \( x^c(t) \) depends on \( c \). Using a Taylor series expansion,

\[
x^c(t) = x^0(t) + \frac{\partial x^c}{\partial c} \mid_{c=0} c + O(c^2) \tag{11}
\]

Thus, for small values of \( c \) such that the change in the optimal path is small, i.e., \( \frac{\partial x^c}{\partial c} \ll 1 \), one has \( x^c(t) \approx x^0(t) \). For small \( c \), the action of the MPSP is therefore given by,

\[
\mathcal{R}^c \approx \frac{1}{2} \int_{-\infty}^{\infty} [\dot{x}^0 - F(x^0) - c f(x^0)]^T [\dot{x}^0 - F(x^0) - c f(x^0)] dt \approx \frac{1}{2} \int_{-\infty}^{\infty} [\dot{x}^0 - F(x^0)]^T [\dot{x}^0 - F(x^0)] dt - c \int_{-\infty}^{\infty} f(x^0)^T \dot{x}^0 dt = \mathcal{R}^0 - c \int_{-\infty}^{\infty} f(x^0)^T \eta^0(t) dt.
\]

This can be written concisely as

\[
\mathcal{R}^c \approx \mathcal{R}^0 - \alpha c \tag{12}
\]

where

\[
\alpha = \int_{-\infty}^{\infty} f(x^0)^T \eta^0(t) dt,
\]

and \( \eta^0(t) \) is the optimal noise profile for the uncontrolled case. Note that similar to \( \eta^0 \), \( f(x) \) is always directed towards the basin boundary, and as such \( f(x^0) \approx \frac{\eta^0}{|\eta^0(t)|} \), i.e., \( f(x^0) \) and \( \eta^0 \) are approximately parallel. In addition, \( \lim_{t \to \pm \infty} \eta^0(t) = 0 \). Thus, \( 0 < \alpha < \infty \), and the change in the action due to the external control field is

\[
\Delta \mathcal{R} = -\alpha c. \tag{13}
\]

Using (5), the change in the mean switching time due to this change in the action is given by

\[
\frac{T_E^c}{T_E^0} = e^{\Delta \mathcal{R}/D}, \tag{14}
\]

where \( T_E^c \) and \( T_E^0 \) are the MST for the controlled and uncontrolled cases respectively. From (13) and (14) it can be seen that \( c < 0 \) implies \( T_E^c > T_E^0 \). Similarly \( c > 0 \) implies \( T_E^c < T_E^0 \). Thus, it is evident that the MST of a system can be changed using an external control signal of the suggested form. If the dynamical system and the noise in the system are completely known, (14) can be used to compute the external control required to achieve a desired MST. However, in practical systems, these details are often unknown. The synthesis of a control strategy to obtain a desired MST in such cases where the details about the dynamical system and/or the noise in the system are not fully known, is presented in the following section.

IV. CONTROL STRATEGY

From (5), it can be seen the average time required to escape from one attractor depends on the action as well as the amount of noise in the system. For a given noise intensity, the MST is governed by the action of the transition path that is most likely to occur. The objective of this work is to use a control of the form \( u = cf(x) \), in which the parameter \( c \in [-c_{\text{max}}, c_{\text{max}}] \) can be varied to obtain a desired MST, \( T_E^0 \), by changing the action.

If the noise intensity \( D \), the current MST \( T_E^0 \), and the dynamics of the system are fully known, then (13) and (14) can be used to compute the value of \( c \) required to obtain a desired MST. Typically, most of this information is not readily available in a real system. Thus, to design a control strategy to obtain the desired MST, we must first understand the characteristic behavior of a noise-driven switching trajectory in a dynamical system with multiple stable states. Figure 3a shows the typical variation of the distance \( d \) between a point on a trajectory and the closest basin boundary over time until escape from the attracting region through one of the basin boundaries occurs. A simplified \( d \) versus \( t \) plot that captures the essential characteristic of the curve in Fig. 3a is shown in Fig. 3b. Although Fig. 3a is generated using a trajectory realization obtained from a double-gyre flow field, this type of variation for the distance to boundary is typical of switching trajectories in general multi-stable dynamical systems. A major portion of the system trajectory is concentrated around the stable equilibrium, before it suddenly transitions out of the attracting region. The actual transition itself occurs over a fraction of the overall dwell time, and near the transition, the \( d \) versus \( t \) curve is approximately linear. These typical characteristics can be used to identify a potential onset of the escape portion of a trajectory, when neither the noise level of the system nor the MSTs are known.
current uncontrolled MST can be computed to be
\[ T_E^0 = \frac{t}{1 - \lambda_t \frac{2d}{s}}. \]  (15)

Using (14), the required change in action to obtain the desired MST can be approximated as,
\[ \Delta R = k \log\left(\frac{T_E^d}{T_E^0}\right), \]
where \( k \) is a user defined parameter which governs how aggressive the control is. Using (13), the control parameter \( c \) is set to be,
\[ c = \max\left( -\frac{\Delta R}{\alpha}, -c_{\text{max}} \right). \]  (16)

When \((t, d) \in R_2\) (e.g., dashed portion of the red trace in Fig. 4), it is assumed that \( T_E^0 \geq T_E^d \), and that the particle has not started its transition towards escape. In contrast to the previous case, an estimate for \( T_E^d \) cannot be obtained. Furthermore, in order to meet the desired MST target, the particle must transition out as soon as possible. In this case, the control parameter is set as
\[ c = c_{\text{max}}. \]

Therefore, the proposed control strategy is based on making local assumptions about \( T_E^d \), and is given by \( u = c f(x) \) where \( c \) is defined as
\[ c = \begin{cases} \max\left( -\frac{\Delta R}{\alpha}, -c_{\text{max}} \right) & (t, d) \in R_1 \\ c_{\text{max}} & (t, d) \in R_2 \\ 0 & (t, d) \in R_3 \end{cases}, \]  (17)
and as shown in section III, \( f(x) \) is a unit vector pointed towards the closest basin boundary.

Essentially, the control strategy pushes the agent away from the basin boundary if it gets close to the boundary before the required amount of time has elapsed, and it pushes the agent towards the boundary when the elapsed time is close to the required MST. The instances at which the control is switched on are governed by the parameters \( \lambda_s \) and \( \lambda_t \). Note that \( \lambda_t \leq 1 \) and \( 0 \leq \lambda_s \leq 1 \). Intuitively it can be seen that large values of \( \lambda_s \) will increase the MST and that large values of \( \lambda_t \) will decrease the MST.

B. Analysis of the control strategy

In order to analyze the proposed control strategy and verify its correctness, the strategy is analyzed using a 1D system. This greatly simplifies the analysis while preserving the essential characteristics of the controlled system. Insights from the 1D system are then used to select values for \( \lambda_s \) and \( \lambda_t \). Consider a particle in a 1D potential well, subject to Gaussian noise. The equation of motion of this particle is given by
\[ \dot{x} = -\frac{\partial U}{\partial x} + \eta(t) + u(t), \]  (18)
where \( x \) is the position, \( U \) represents the potential well (see Fig. 6), \( u(t) \) is the control, and \( \eta \) is Gaussian noise with intensity \( D \). For the uncontrolled case, i.e., \( u(t) = 0 \), it has been shown\(^{21-23} \) that if \( \Delta U/D \gg 1 \), the average time \( T_E \) required for a particle to escape the stable equilibrium at \( x_{\text{min}} \) is given by

\[
T_E^0 = \frac{1}{D} \int_{x_1}^{x_2} e^{-\left( \frac{U_{\text{min}}}{D} + \frac{U''_{\text{min}}}{2D}(x-x_{\text{min}})^2 \right)} \, dx
\]

where \( U''_{\text{min}} \) and \( U''_{\text{max}} \) are the second derivatives of \( U(x) \) at \( x_{\text{min}} \) and \( x_{\text{max}} \) respectively, and \( A \) is a point away from \( x_{\text{max}} \) as shown in Fig. 6. Further details of this derivation can be found in a recent review article\(^{23} \). Considering the exponential fall off of the integrands, and extending the limits of both integrals from \(-\infty \) to \( \infty \), one can show that

\[
T_E^0 = \frac{2\pi}{\sqrt{|U''_{\text{min}}|/|U''_{\text{max}}|}} e^{\frac{\Delta U}{D}}.
\]

This is the well known Kramers’ escape rate for 1D systems\(^{14} \).

Now consider a control of \( u(t) = c \partial U / \partial x \) with \( |c| \leq c_{\text{max}} < 1 \). Such a control results in a controlled 1D system given by

\[
\dot{x} = -(1-c) \frac{\partial U}{\partial x} + \eta(t).
\]

This is equivalent to considering a potential well \( \tilde{U} = (1-c)U \). Thus, for \( c < 0 \), the well becomes deeper and for \( 0 < c < 1 \) the well becomes shallower. Substituting \( U = \tilde{U} \) in (19) and (20), one obtains

\[
T_E^c = \frac{2\pi}{(1-c)\sqrt{|U''_{\text{min}}|/|U''_{\text{max}}|}} e^{(1-c)\frac{\Delta U}{D}} = e^{-\frac{\Delta U}{1-c}T_E^0}.
\]

It can be shown that \( c < 0 \Rightarrow T_E^c > T_E^0 \), and \( 0 < c < 1 \Rightarrow T_E^c < T_E^0 \) when \( \Delta U/D \gg 1 \), i.e., \( c > 0 \) pushes the particle out towards the boundary and \( c < 0 \) pulls the particle in towards the center of the well.

The corresponding region based control strategy for \( c \) as proposed in (17) is given by

\[
c \begin{cases} < 0, & x_s \leq x < x_{\text{max}} \text{ and } t < T_t \\ > 0, & t \geq T_t \\ = 0, & \text{otherwise} \end{cases}
\]

where \( x_s = x_{\text{max}} - \lambda_s(x_{\text{max}} - x_{\text{min}}) \) and \( T_t = (1 - \lambda_t)T_E^0 \), with \( \lambda_t \leq 1 \) and \( 0 \leq \lambda_s \leq 1 \). In the remainder of this section, we show that this region based controller is able to achieve any desired MST within bounds that are dependent on the maximum available control.

To obtain an expression for the MST under the proposed control strategy, first consider applying a control with \( c < 0 \) for \( x_s \leq x < x_{\text{max}} \), without considering the elapsed time (case 1 of the control strategy in (23)). In this case, the control action can be written as \( u(t) = -|c|\theta(x - x_s) - \theta(x - x_{\text{max}})\frac{\Delta U}{D} \), where \( \theta \) is a Heaviside function. Thus, the first integral \( I_1 \) of (19) for the mean escape time, can now be written as,

\[
I_1 = \int_{x_s}^{x_2} e^{-\left( \frac{U_{\text{min}}}{D} + \frac{U''_{\text{min}}}{2D}(x-x_{\text{min}})^2 \right)} \, dx
\]

\[
+ \int_{x_2}^{x_{\text{max}}} e^{-\left(1+|c|\right)\left( \frac{U_{\text{min}}}{D} + \frac{U''_{\text{max}}}{2D}(x-x_{\text{max}})^2 \right)} \, dx.
\]

Considering that the integrands of both integrals decay exponentially, the lower limit of the first integral can be extended to \(-\infty \), and the upper limit of the second integral can be extended to \( \infty \). Thus,

\[
I_1 = \sqrt{|U''_{\min}|/|U''_{\max}|} e^{\frac{U_{\text{min}}}{D}} \left( 1 + \text{erf}\left( \sqrt{\frac{|U''_{\min}|}{2D}} (x_s - x_{\text{min}}) \right) \right)
\]

\[
+ e^{-\frac{|c|U_{\min}}{\sqrt{1+|c|}}} \left( 1 - \text{erf}\left( \sqrt{\frac{(1+|c|)U''_{\min}}{2D}} (x_s - x_{\text{min}}) \right) \right).
\]

Using similar arguments, the second integral \( I_2 \) of (19) can be written as

\[
I_2 = \sqrt{|U''_{\max}|/|U''_{\min}|} e^{\frac{U_{\text{max}}}{D}} \left( 2 + \text{erf}\left( \sqrt{\frac{|U''_{\max}|}{2D}} (x_s - x_{\text{max}}) \right) \right)
\]

\[
- e^{-\frac{|c|U_{\max}}{\sqrt{1+|c|}}} \text{erf}\left( \sqrt{\frac{(1+|c|)U''_{\max}}{2D}} (x_s - x_{\text{max}}) \right).
\]

In (25) and (26), \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \). Thus, the expected MST, when control of \( c < 0 \) is enacted for \( x_s \leq x < x_{\text{max}} \) is

\[
T_E^c_{\text{dist}} = \frac{1}{D} I_1 I_2.
\]

Next, consider introducing a control with \( c > 0 \) when \( t \geq T_t \). Due to the stochastic nature of escape events, of the total paths that escape, \( \int_0^{T_t} P_{E\theta}(t) \, dt \) of them would have already escaped before the \( c > 0 \) control is switched on at \( t = T_t \). Recalling (7), we know that \( P_{E\theta}(t) \) is the probability distribution of the escape times before switching on the \( c > 0 \) control, and it is exponentially distributed, i.e.,

\[
P_{E\theta}(t) = \frac{1}{T_E} e^{-t/T_E}.
\]

The percentage of particles escaping after turning on the \( c > 0 \) control is \( 1 - \int_0^{T_t} P_{E\theta}(t) \, dt \), and the mean escape time for these particles is \( T_{c > 0} = T_E + T_c \), where
$T_E$ is the mean escape time if the $c > 0$ control is applied \( \forall t \geq 0 \), and is given by (22). Thus the expected mean switching time under the full control strategy proposed in (23) is

$$T_E^{exp} = \int_0^T t P_{TE}(t)dt + (T_i + T_E)\left(1 - \int_0^T P_{TE}(t)dt\right).$$

Using (7), this can be simplified as

$$T_E^{exp} = T_E^{dist} - (T_E^{dist} - T_E)\exp\left(\frac{-\lambda T_E^d}{T_E^{dist}}\right), \quad (28)$$

where $T_E^{dist}$ is given in (27), and $T_E$ is the nominal mean switching time if a control of $c > 0$ is used \( \forall t \geq 0 \). Note that we have also used the fact that $T_i = (1 - \lambda t)T_E^d$, where $T_E^d$ is the desired MST.

For $c = 0$, it is trivial to see that $T_E^c = T_E^{dist} = T_E^d$. It can be shown that for $\Delta U/D \gg 1$, $\partial T_E^{dist}/\partial c > 0$ and $\partial T_E^d/\partial c < 0$. Thus, it can be inferred that $T_E^{dist} \geq T_E^d$, with equality at the trivial case of $c = 0$. Using this, one can easily show that $\frac{\partial T_E^{exp}}{\partial c} < 0$ and is continuous for $\lambda \leq 1$. Thus, $T_E^{exp}$ is minimized at $\lambda = 1$ and it is maximized as $\lambda \rightarrow -\infty$. Thus, from (28), it can be seen that

$$T_E^c \leq T_E^{exp} < T_E^{dist}, \quad (29)$$

with $T_E^{exp} = T_E^c$ for $\lambda = 1$, and $T_E^{exp} \rightarrow T_E^{dist}$ for $\lambda \rightarrow -\infty$. Thus, for a given $c$ and $\lambda$, there exists $\lambda_1 \leq 1$ that can achieve a desired escape time in the range established in (29).

As mentioned before, it can be shown that for $\Delta U/D \gg 1$, $\partial T_E^{dist}/\partial |c| > 0$ and $\partial T_E^{dist}/\partial \lambda | > 0$. In addition, $T_E^{dist}\mid_{c=0} = T_E^d$ and $T_E^{dist}\mid_{\lambda=0} = T_E^c$. Thus, the maximum of $T_E^{dist}$ occurs at $c = -c_{max}$ and $\lambda_s = 1$. Substituting these values in (27), we have

$$T_E^d \leq T_E^{dist} \leq T_{max}, \quad (30)$$

where

$$T_{max} = \frac{T_E^d}{4} \left(1 + \frac{e^{-c_{max} U_{min} D}}{\sqrt{1 + c_{max}}}\right)\left(1 + \frac{e^{c_{max} U_{max} D}}{\sqrt{1 + c_{max}}}\right).$$

Thus there exists a $(-c_{max} \leq c \leq 0, 0 \leq \lambda_s \leq 1)$ tuple that can achieve any $T_E^{dist}$ value in the range given in (30).

In a similar fashion, one can show that there exists a $0 \leq c \leq c_{max}$ that can achieve any $T_E^c$ value in the range,

$$T_{min} \leq T_E^c \leq T_E^d$$

where

$$T_{min} = \frac{e^{-c_{max} U_{min} D}}{1 - c_{max}} T_E^d.$$

From the above observations it can be concluded that there exist $-c_{max} \leq c \leq c_{max}, \lambda_s \leq 1$ and $0 \leq \lambda_s \leq 1$ that can achieve any desired mean switching time in the range $T_{min} \leq T_E^c < T_{max}$.

It is worth noting that for the above controller, $c < 1$ was considered in the analysis. If $c > 1$, the peak and the trough of the effective potential $(1 - c)U$ will be swapped, and the $\Delta U/D \gg 1$ assumption would not hold anymore.

C. Controller Parameter Selection in for general systems

If $T_E^d$ lies between the $T_{min}$ and $T_{max}$ limits specified previously, there always exists a set of $(c, \lambda_t, \lambda_s)$ values that will achieve the desired MST. If the noise intensity $D$ is known, depending on $T_E^d$, a suitable set of $(c, \lambda_t, \lambda_s)$ values can be selected to achieve $T_E^d$. In general, the noise level $D$ is not known. In such cases, not only is it impossible to determine a set of $(c, \lambda_t, \lambda_s)$ values to achieve a given $T_E^d$, but it is also not possible to determine if the required $T_E^d$ value is even feasible. In a general higher-dimensional system, selecting a set of $(c, \lambda_t, \lambda_s)$ is even more complicated since an expression for $T_E$ of the form given in (22) is not available.

In the control strategy given in Sec. IV A, the problems outlined above are overcome by first selecting values for $c, \lambda_t, \lambda_s$ that approximately achieve the desired MST for $T_E^d > T_E^c$, and then by refining $\lambda_t$ to achieve $T_E^d$. For the $T_E^d > T_E^c$ case, the current uncontrolled MST $T_E^d$ is approximately estimated using (15), and then a value for $c$ that would make $T_E^d \rightarrow T_E^c$ is selected using $c < 0$ control alone. Note that for this $T_E^d$ to be achieved using $c < 0$ control alone, $\lambda_t = 1$. Thus, for this case $T_E^{dist} \approx T_E^d$. According to (28), to make $T_E^{exp} \approx T_E^{dist} \approx T_E^d$, we need $\lambda_t \rightarrow -\infty$. That is, by selecting a large value for $\lambda_t$ and a large negative value for $\lambda_s$, we are able to approximately achieve $T_E^d$ if $T_E^d > T_E^c$. However, if $T_E^d < T_E^c$, this large negative value for $\lambda_t$ will not be able to achieve the required $T_E^d$.

Thus, in order to achieve the desired MST for both $T_E^d > T_E^c$ and $T_E^d < T_E^c$, we select $0 \leq \lambda_t < 1$ and $0 < \lambda_s < 1$, i.e., $\lambda_s$ close to 1, and $\lambda_t$ close to zero.

V. RESULTS

The control strategy given in Sec. IV A was used to control the mean switching time to a desired value in two dynamical systems exhibiting multiple stable equilibria, the double-gyre flow model and the damped pendulum model. In all of the following simulations, the Euler-Maruyama method was used for integrating the stochastic differential equations.

A. Simulation results for a double-gyre flow

The double-gyre flow model is often used to describe large scale recirculation in the ocean\textsuperscript{17}, and it is given in (6). Figure 1a shows the phase portrait of the flow for $A = 1$, $s = 1$, and $\mu = 1$. For $\mu > 0$, each gyre has a deterministic attractor in the center of the gyre, and is flanked by four saddle points. The gyre boundaries consist of the stable and unstable manifolds of these saddle points. A system of two adjoining gyres as shown in Fig. 1a, qualitatively resembles a double-well potential.

Using the same parameter values ($A = 1, \mu = 1$ and $s = 1$), the stochastic double-gyre was considered in the simulations, with noise intensities given in the set $D = \{1/30, 1/40, 1/50, 1/60, 1/70\}$. For each noise intensity, a set of desired MSTs given by $T_E^d = \{3, 6, 12, 26, 57, 122, 262\}$ were considered. These $T_E^d$ values approximately correspond to the natural mean switching times $T_E^d$ for noise intensities $\{1/20, 1/30, 1/40, 1/50, 1/60, 1/70, 1/80\}$ respectively. For each $(D, T_E^d)$ pair, 1000 simulation trials, each starting near
Controlling Stochastic Transition Rates

FIG. 7. Desired MSTs $T_E^{d}$ versus the actual MSTs $T_E^{act}$ for different $(\lambda_s, \lambda_t)$ value combinations. The values of $\lambda_s$ and $\lambda_t$ for each set is given in Table I.

![Graphs for different sets showing desired and actual MSTs](image1)

TABLE I. Values of $\lambda_s$ and $\lambda_t$ used in the simulations.

<table>
<thead>
<tr>
<th></th>
<th>set1</th>
<th>set2</th>
<th>set3</th>
<th>set4</th>
<th>set5</th>
<th>set6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_s$</td>
<td>0.85</td>
<td>0.85</td>
<td>0.85</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>$\lambda_t$</td>
<td>-1</td>
<td>0.0625</td>
<td>0.9</td>
<td>-1</td>
<td>0.0625</td>
<td>0.9</td>
</tr>
</tbody>
</table>

...the center of the left gyre, were conducted. Each simulation trial was terminated when the system state escaped the boundaries of the left gyre. In all simulations, $c_{max} = 0.5$ was considered. In order to investigate the effect of selecting different $\lambda_s$ and $\lambda_t$ values, simulations were run for the $(\lambda_s, \lambda_t)$ value combinations given in Table I. In sets 1-3, a large value is selected for $\lambda_s$ ($\approx 1$), and in sets 4-6, a small value is selected for $\lambda_s$. In both cases, $\lambda_t$ is successively increased from a negative value towards 1.

From the discussion in Sec. IV C, the best results should be expected for set 2, where $\lambda_s$ is large and $\lambda_t$ is moderate. Figures 7a-7f show plots of desired MST, $T_E^{d}$, versus the actual MST, $T_E^{act}$, for different $(\lambda_s, \lambda_t)$ value combinations. For each set of $(\lambda_s, \lambda_t)$ values, multiple noise levels are considered. In each figure, the thick dotted line in black represents the ideal $T_E^{d} = T_E^{act}$ curve. The closer the $T_E^{d}$ versus $T_E^{act}$ curves are to this line, the better the performance of the control strategy.

Figures 7a-7f show that set 2 ($\lambda_s = 0.85, \lambda_t = 0.0625$), indeed gives the best results. In set 1, $T_E^{act}$ overshoots $T_E^{d}$ by a considerable margin since the negative value used for $\lambda_t$ cannot pull back $T_E^{act}$ dist in (28) enough towards $T_E^{d}$. On the other hand, in set 3, where $\lambda_t$ is close to 1, $T_E^{act}$ is pulled too far back by the $c > 0$ control, which results in very small $T_E^{act}$ values. Sets 4 and 6 follow similar behaviors as set 1 and set 3 respectively due to the effect of $\lambda_t$. While sets 2 and 5 consider the same moderate value for $\lambda_t$, in set 5, $T_E^{act}$ undershoots $T_E^{d}$ due to the small value of $\lambda_s$ considered in set 5. Note that even in set 2, which has the best performance, large desired MSTs cannot be obtained when the noise in the system is high (see red line in Fig. 7b). In such cases, the available control is not sufficient to achieve MSTs which are much greater than the "natural" MST of the system. In these cases the desired MST falls outside the established limits.

Figures 8a-8f show the probability densities of the MSTs obtained for various values $T_E^{d}$ for a noise level of $D = 1/60$, where $\lambda_s = 0.85$ and $\lambda_t = 0.0625$ were used for the control.

![Probability densities for different MSTs](image2)
seen that the control strategy proposed in Sec. IV A is able to achieve a wide range of desired MSTs, for a wide range of system noise levels.

The control strategy also was tested with a non-Gaussian noise source to check its performance in a non-ideal scenario. In this case the noise signal was derived as \( \eta(t) = \tilde{\sigma}z^{1/3}(t) + \delta \) where each component of \( z \) has a standard normal distribution, i.e., \( z_i \sim \mathcal{N}(0, 1) \). The value for \( \tilde{\sigma} \) was selected so that the standard deviation of each component of \( \eta \) was equal to the standard deviations considered in the Gaussian case, i.e., \( \sigma(\eta_i) = \sqrt{2D} \) for \( i = 1, 2 \). As each component of the mean \( \delta \) is the same value \( \delta \), this term essentially shifts the flow velocities in (1) by a constant amount, and its value is selected to be small enough such that the gyre structure of the flow is maintained. Figure 9 shows the results for the case where \( \delta = 0.1 \), \( \lambda_s = 0.85 \) and \( \lambda_t = 0.0625 \). In the cases shown \( \tilde{\sigma} \) was selected such that the noise signals have the same standard deviations as before. It can be seen that the desired MSTs are achieved even in the presence of non-Gaussian noise sources.

B. Simulation results for a damped pendulum

The proposed control method was used to control the MST in a damped pendulum system given by

\[
\ddot{\theta} = -\frac{g}{L} \sin \theta - \beta \dot{\theta} + u, \tag{31}
\]

where \( \theta \) is the angle measured anti-clockwise from the downward direction, \( g \) is the gravitational constant, \( L \) is the length of the pendulum, \( \beta \) is the damping coefficient and \( u \) is the external control force. Considering the state space representation of this system, the noise-affected system can be expressed in the form of (8) where,

\[
\mathbf{F}(x) = \begin{bmatrix}
\omega \\
-\frac{2}{3} \sin \theta - \beta \omega
\end{bmatrix}, \tag{32}
\]

with state \( x = [\theta, \omega]^T \), control \( u = [0, u]^T \) and noise \( \eta = [0, \eta]^T \). Note that 1D control and noise fields are considered since the original system only has a 1D control.

A damped pendulum with parameters \( L = 1 \) and \( \beta = 0.1253 \) was considered in the simulations (see Fig. 10). Noise intensities in the set \( D = \{1/3, 1/2, 1\} \) were used in the simulations, and for each noise intensity, a set of desired MSTs given by \( T^d_E = \{1778, 254, 45, 25\} \) were considered. These \( T^d_E \) values approximately correspond to the natural mean switching times \( T^u_E \) for noise levels \( \{1/3, 1/2, 1, 1.5\} \) respectively. For each (D, \( T^d_E \)) pair, 1000 trials were simulated until escape through the basin boundary (the basin boundary is shown by the black line in Fig. 10). As proposed in section IV C, a large value was selected for \( \lambda_s \) and a small value was selected for \( \lambda_t \). Figure 11 shows the desired MST \( T^d_E \) versus the actual MST \( T^u_E \) for different noise intensities with \( \lambda_s = 0.9 \) and \( \lambda_t = 0.01 \). It can be seen that the proposed external control is able to achieve MSTs that are much different to the natural uncontrolled switching time of the system.

VI. CONCLUSIONS

In this work a control strategy that could be used to control the mean switching time (MST) in a multi-stable dynamical system affected by external noise was presented. The main idea was to use an external control signal to obtain a desired MST. It was shown that the control strategy could be used to enhance or abate the MST by changing the action of the noise required to affect a transition. A specific controller, inspired by the most probable noise profile leading to transition, was proposed to control the MST to a desired value. The controller was analyzed in a 1D system and it was
shown that the controller can achieve any MST in a bounded interval whose limits are governed by the amount of control actuation available. The strategy was evaluated in simulations using two dynamical systems, the double-gyre flow and the damped pendulum. The results show that the controller is indeed able to obtain desired MSTs for various noise levels in the system including non-Gaussian noise.

ACKNOWLEDGMENTS

This work was funded by the Office of Naval Research (ONR) grant N000141712690 and the National Science Foundation (NSF) awards CMMI-1418956, CMMI-1462884, DMS-1418956, and IIS-1253917.


